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Residual Cutting Method for Elliptic Boundary Value Problems:

Application to Poisson's Equation

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A new, efficient, and highly accurate numerical method which achieves the residual reduction with the aid of residual equations and the method of least squares is proposed for boundary value problems of elliptic partial differential equations. Neumann, Dirichlet, and mixed boundary value problems of three-dimensional Poisson's equation for pressure in a curved duct flow and a cascade flow have been solved. Numerical results exhibit the effectiveness of the method by a high convergence rate and a high degree of robustness. The method is expected to be an effective numerical solution method applicable to a wide range of partial differential equations with various boundary conditions. © 1997 Academic Press

Key Words: computational fluid dynamics; numerical method; boundary value problem; Poisson's equation; Neumann problem.

1. INTRODUCTION

In most numerical approaches, boundary value problems of elliptic partial differential equations are discretized and reduced to finite difference equations, after which linear systems with huge matrices are solved using appropriate numerical procedures. However, the numerical methods currently available, such as SOR, CG or BCG, ADI, and the multigrid method with SOR etc. seem to be not accurate enough and the rate of convergence is not necessarily high for three-dimensional actual engineering applications, especially for Neumann problems of Poisson's equation [1–7].

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It is recognized that the three-dimensional Neumann problem still remains a major difficulty in general curvilinear coordinate systems. The difficulty involved is due to the characteristics of corresponding matrices and a compatibility condition (an integral constraint) which relates the source term of the equation and boundary conditions of the second kind, and these lower both the rate of convergence and the level of robustness.

The purpose of the work reported here is to propose an efficient and accurate numerical method called herein the residual cutting method for boundary value problems of elliptic partial differential equations.

This numerical method is intended to get approximate solutions with a smaller residual L2 norm. The procedure of the residual cutting method is described as follows. Using the residual r^m corresponding to the approximate solution U^m at mth step, we set a residual equation. We solve the residual equation within the first few iterations to get a roughly approximated solution ψ^m. We then define the perturbed quantity ϕ^m which is the linear combination of ψ^m and previous ϕ^{m-1}, ϕ^{m-2}, ... The coefficients are determined by the method of least squares so as to reduce the residual L2 norm in the sense that $\|r^{m+1}\| < \|r^m\|$. Then, a new approximate solution U^{m+1} and a new residual r^{m+1} are obtained using the resultant perturbed quantity ϕ^m. The residual equation is updated and the above procedure is used in an iterative manner. Ultimately we get the target solution and the residual. The final residual becomes zero for well-posed Neumann, Dirichlet, and mixed boundary value problems, but for an ill-posed Neumann problem, the final residual L2 norm will only become a certain minimum that is not zero.

Roughly speaking, in the residual cutting method the "residual reduction" is achieved at the expense of extra inner product operations, instead of the "error reduction" in projective type methods like CG and GMRES [8]. In fact, it is shown that $A \cdot \phi^{m-1}$, $A \cdot \phi^{m-2}$, ... and r^m are orthogonal to each other. Thus our method provides with a new aspect of iteration method and, hence, it is interesting to compare theoretically our method with popular iteration methods as mentioned above. We also remark that it may be possible to adopt other iteration methods as an inner iteration. This exhibits another difference of our method from well-known methods including the multigrid method. The mathematical analysis on the residual cutting method will be published elsewhere.

To show the validity of the present method, three-dimensional Poisson's equations for pressure, in both orthogonal curvilinear and general curvilinear coordinate systems, are successfully solved for the Neumann problem, as well as for the Dirichlet and mixed boundary value problems. Numerical results show that the residual cutting method has many desirable characteristics such as robustness, accuracy and efficiency in actual applications.

2. PROCEDURE OF RESIDUAL CUTTING METHOD

Throughout this paper we denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and L2 norm in R^n, respectively.

Generally, in order to solve boundary value problems of elliptic partial differential equations numerically, which are discretized into finite difference equations, it is needed to solve the linear system of n equations given in (1):

$$A \cdot U = f. \tag{1}$$

Now let U^m be an approximate solution of (1). The corresponding residual r^m is defined as (2):

$$r^m = f - A \cdot U^m. \tag{2}$$

Let the solution U^∞ of (1) be given as the sum of U^m and ϕ as in (3):

$$U^\infty = U^m + \phi. \tag{3}$$

Substitute (3) into (1), and using (2) we obtain the residual equation given in (4):

$$A \cdot \phi = r^m. \tag{4}$$

In the usual procedure, residual equation (4) is solved up to convergence in an iterative manner. However, in the residual cutting method, the most important objective is to get information from (4) which will reduce the residual r^m, not to obtain a solution of (4). That is to say, an approximate solution ψ^m of (4) is obtained within the first few iterations and is used to construct the next U^{m+1} and r^{m+1}.

We are now in a position to describe the procedure of the residual cutting method. Let $m \geq 0$ and $1 \leq L \leq m + 1$. Assume that U^m, r^m, ψ^m, and ϕ^j $(-1 \leq j \leq m - 1)$ are given, where $\phi^{-1} = 0$. The perturbed quantity ϕ^m is defined as (5) and the approximate solution U^{m+1} is defined as (6):

$$\phi^m = \alpha_1^m \psi^m + \sum_{l=2}^{L} \alpha_l^m \phi^{m-l+1} \tag{5}$$

$$U^{m+1} = U^m + \phi^m. \tag{6}$$

Here the second term of the right-hand side in (5) is taken to be zero for $L = 1$. The corresponding residual r^{m+1} is represented by (7):

$$r^{m+1} = r^m - A \cdot \phi^m = f - A \cdot U^{m+1}. \tag{7}$$

Each of the real numbers α_l^m $(l = 1, 2, 3, ..., L)$ in (5) is called a residual cutting coefficient, and determined so as to minimize the value $\|r^m - \alpha_1 A \cdot \psi^m - \sum_{l=2}^{L} \alpha_l A \cdot \phi^{m-l+1}\|$. In order to determine the residual cutting coefficients α_l^m, the method of least squares is used, that is, the linear system of L equations obtained from (8) is solved:

$$\frac{\partial}{\partial \alpha_l} \left\| r^m - \alpha_1 A \cdot \psi^m - \sum_{l=2}^{L} \alpha_l A \cdot \phi^{m-l+1} \right\|^2 = 0, \quad l = 1, 2, 3, ..., L. \tag{8}$$

Making full use of the facts that the least squares matrix is symmetric and $A \cdot \phi^{m-1}$,

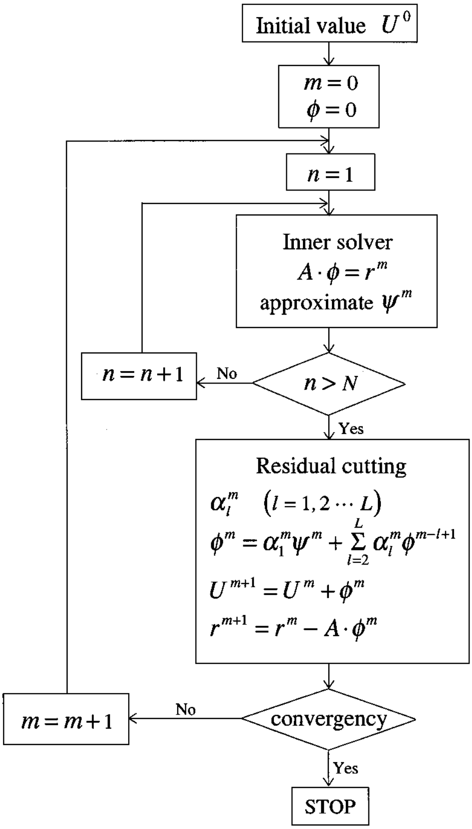


FIG. 1. Flowchart of residual cutting method.

$A \cdot \phi^{m-2}$, ..., and r^m are orthogonal to each other (see the next section), α_l^m are simply determined.

To solve residual equation (4) approximately, the ADI method is adopted because it is more suitable for parallel vector computers, and solutions are obtained by directly using the 3-term recurrence relation. We here recommend that zero be set as the initial value in the ADI iteration.

The procedural flow of the residual cutting method is (see Fig. 1)

(1) Give an initial value U^0 and calculate $r^0 = f - A \cdot U^0$.

(2) Solve $A \cdot \phi = r^m$ to get ψ^m within only a very few iterations.

(3) Calculate residual cutting coefficients α_l^m ($l = 1, 2, 3, ..., L$).

(4) Calculate $\phi^m = \alpha_1^m \psi^m + \sum_{l=2}^{L} \alpha_l^m \phi^{m-l+1}$.

(5) Calculate $U^{m+1} = U^m + \phi^m$ and $r^{m+1} = r^m - A \cdot \phi^m$ $(=f - A \cdot U^{m+1})$.

Repeating (2) to (5) until U^m converges, we obtain the solution U^∞ with $r^\infty = 0$ for well-posed Neumann, Dirichlet, and mixed boundary value problems. On the other hand, for an ill-posed Neumann problem, we obtain a certain quantity U^∞ with the residual $r^\infty \neq 0$ (see Section 4). It should be noted that $\|r^{m+1}\| < \|r^m\|$ if and only if $(r^m, A \cdot \psi^m) \neq 0$. In view of this, the residual cutting method is interpreted as the residual reduction method and it will be clear that the property of the method is quite different from those of CG etc.

It is also mentioned that the number L for α_l^m seems to be enough with $3 \leq L \leq 6$ to solve most problems (see Section 6). It is noticed that the residual cutting method needs $L + 1$ inner products per step. (When $L = 1$, the number of new inner products is equal to that of the CG method.) Although this may be a time-consuming process, it is emphasized that high convergence rates are achieved at the expense of those operations.

3. ORTHOGONALITY AND CONVERGENCE

In this section U^m, r^m, ψ^m, and ϕ^m denote the vectors in the residual cutting method.

PROPOSITION 1 (Orthogonality). *Let $m \geq 0$ and suppose that $(r^m, A \cdot \psi^m) \neq 0$. Then we have*

$$(r^{m+1}, A \cdot \phi^j) = 0, \quad m + 1 - L \leq j \leq m, 1 \leq L \leq m + 1; \tag{9}$$

$$(A \cdot \phi^j, A \cdot \phi^k) = 0, \quad m + 1 - L \leq j, k \leq m, j \neq k, 2 \leq L \leq m + 1. \tag{10}$$

Proof. It follows from the method of least squares that $(r^{m+1}, A \cdot \psi^m) = 0$ and $(r^{m+1}, A \cdot \phi^j) = 0$ for $\min\{m - 1, m + 1 - L\} \leq j \leq m - 1$ $(1 \leq L \leq m + 1)$, where $\phi^{-1} = 0$. Since ϕ^m is the linear combination of ψ^m and ϕ^j, it is obvious that $(r^{m+1}, A \cdot \phi^m) = 0$. Hence (9) is proved. To prove (10), let $m \geq 1$ and $2 \leq L \leq m + 1$. By solving the problem of least squares, we have that

$$(A \cdot \phi^m, A \cdot \phi^j) = (r^m, A \cdot \phi^j) = 0, \quad m + 1 - L \leq j \leq m - 1.$$

Here we used relation (9) with $m + 1$ replaced by m. This completes the proof. Q.E.D.

At this point we mention that in the GMRES algorithm ϕ^m is determined so as to achieve orthogonality in the following sense:

$$(\phi^j, \phi^k) = 0, \quad 0 \leq j, k \leq m, j \neq k.$$

PROPOSITION 2 (Convergence). *Let $1 \leq L \leq m + 1$ be fixed. Suppose that A is nonsingular, $r^m \neq 0$ for $m \geq 0$, and there is a constant $0 < \theta < 1$, independent of m, such that*

$$\|r^m - A \cdot \psi^m\| \leq \theta \|r^m\|, \quad m \geq 0. \tag{11}$$

Then U^m converges to the target solution U^∞ as $m \to \infty$.

Proof. As is easily checked, the method of least squares assures that

$$\|r^{m+1}\| \le \|r^m - A \cdot \psi^m\|.$$

Combining this with (11), we obtain that

$$\|r^{m+1}\| \le \theta \|r^m\|,$$

which means that $r^m \to 0$ as $m \to \infty$. Since $r^m = f - A \cdot U^m$ and A is nonsingular, it is proved that U^m converges to the target solution U^∞ as $m \to \infty$. Q.E.D.

We notice that condition (11) is satisfied for boundary value problems of the three-dimensional Poisson's equation in a cubic domain. In fact, if we use the standard finite difference equation (the 7-point difference scheme), then the corresponding coefficient matrix A becomes the sum of three matrices which are mutually commutative. By the commutativity, it will be an elementary exercise to check condition (11) [9]. The remark mentioned here suggests that the convergence of U^m may be also affirmative in general case.

4. COMPATIBILITY CONDITION IN THE NEUMANN PROBLEM

It is well known that in the Neumann problem of Poisson's equation, the existence of a solution requires the satisfaction of a compatibility condition (an integral constraint) [5, 6] which relates the source of the equation and boundary conditions of the second kind. The compatibility condition is a consequence of the divergence theorem (Green's theorem).

Now we assume that A is symmetric, positive semidefinite, and rank $(A) = n - 1$. We further assume that $\mathbf{1} = (1, 1, ..., 1)^t$ is the eigenvector corresponding to the minimal eigenvalue $\lambda = 0$ of A. We then have

$$(A \cdot U, \mathbf{1}) = 0 \quad \text{for all } U, \tag{12}$$

which is regarded as a matrix version of the divergence theorem. In the form of Eq. (1), the compatibility condition is expressed as (13):

$$(f, \mathbf{1}) = 0. \tag{13}$$

It is evident that condition (13) is a necessary and sufficient condition for Eq. (1) to possess a solution [10]. For convenience, if $(f, \mathbf{1}) = 0$, Eq. (1) is said to be well-posed and if $(f, \mathbf{1}) \neq 0$, Eq. (1) is said to be ill-posed (not well-posed).

In usual numerical methods, it is necessary to fulfill condition (13), for instance, by modifying the source. On the other hand, in the residual cutting method, condition (13) is not necessarily required, because the method is also applicable to ill-posed problems. In fact, $\|r^m\|$ certainly decreases and it is seen that U^m and r^m converge to certain U^∞ and r^∞, respectively. In the case of ill-posed problem, it turns out that $r^\infty \neq 0$. Furthermore, if the computation is executed by modifying the right-hand side as shown in (14),

$$A \cdot U = f - r^{\infty}, \tag{14}$$

then the same solution U^{∞} as the original ill-posed problem is obtained. This also justifies the numerical procedure proposed in [6]. Thus the residual cutting method is advantageous enough for practical use, and the Neumann problem can be actually dealt with in the same manner as the Dirichlet problem.

5. APPLICATION TO POISSON'S EQUATION

In order to examine the validity of the residual cutting method, several numerical computations are carried out for Neumann, Dirichlet, and mixed boundary value problems of three-dimensional Poisson's equation for pressure.

Poisson's equation for pressure p can be written in the coordinate system (ξ_1, ξ_2, ξ_3) as in (15):

$$\sum_{i,j=1}^{3} \frac{\partial}{\partial \xi_i}\left(J g^{ij} \frac{\partial p}{\partial \xi_j} \right) = S. \tag{15}$$

Here J is the Jacobian of the Cartesian coordinate system (x_1, x_2, x_3) with respect to (ξ_1, ξ_2, ξ_3) and g^{ij} is defined as in (16) and (17), respectively:

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)} \tag{16}$$

$$g^{ij} = \sum_{k=1}^{3} \frac{\partial \xi_i}{\partial x_k} \frac{\partial \xi_j}{\partial x_k}, \quad i, j = 1, 2, 3. \tag{17}$$

The problems to be solved are shown in Table I. In case a the pressure distribution in a curved duct is computed as a fundamental test and in case b the pressure distribution in a turbomachinery cascade is computed to verify the applicability

TABLE I
Test Problems of Three-Dimensional Pressure Field

	Neumann problem		Dirichlet problem	Mixed boundary value problem
	Well-posed	Ill-posed		
Curved duct flow orthogonal curvilinear coordinate system $51 \times 37 \times 51$	a − 1	a − 2	a − 3	
Cascade flow general curvilinear coordinate system $91 \times 47 \times 41$	b − 1	b − 2	b − 3	b − 4

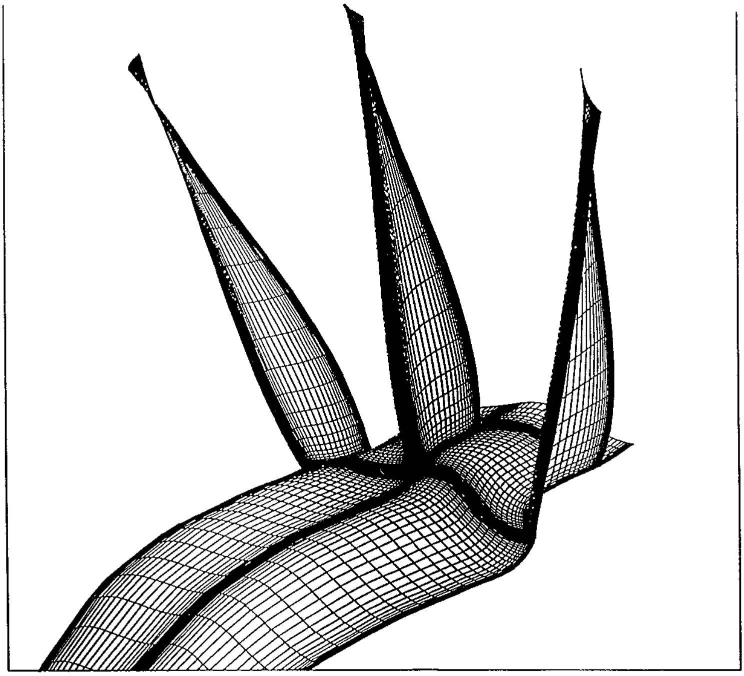


FIG. 2. Computational grid for a cascade.

and robustness of our method. In the mixed boundary value problem b-4, boundary conditions of the first kind are set on two surfaces and boundary conditions of the second kind are set on the other four surfaces. The coordinate systems used for case a and case b are the orthogonal curvilinear and general curvilinear coordinate systems, respectively.

Both of computational grids are nonstaggered and the computational grid for a cascade is shown in Fig. 2. Equation (15) is discretized using the standard central differencing. Namely, the 7-point and 19-point difference equations are respectively obtained for the orthogonal curvilinear and general curvilinear coordinate systems.

We apply the residual cutting method with U replaced by p. It is noted that the right-hand side f in Eq. (1) is composed of the discretization of the source S in Eq. (15) and boundary conditions.

In all problems in Table I we set $p^0 = 0$ at interior grid points as a very rough initial value. In this case it is seen that $r^0 = f$. To solve residual equations, the ADI method of Wachspress–Habetler type [11] is used. In each sweep of the ADI procedure, difference terms corresponding to cross-derivatives are simply handled as terms of the right-hand side, because our main interest lies in robustness.

For convenience, relative residual ε^m and residual cutting ratio δ^m at mth step are defined as

$$\varepsilon^m = \frac{\|r^m\|}{\|f\|} \tag{18}$$

$$\delta^m = \frac{\|r^m\| - \|r^{m+1}\|}{\|r^m\|}. \tag{19}$$

The residual cutting ratio δ_{in}^m in inner ADI is also defined as in (20) (see Fig. 6):

$$\delta_{\text{in}}^m = \frac{\|r^m\| - \|r^m - A \cdot \psi^m\|}{\|r^m\|}. \tag{20}$$

The severe convergence criterion as given in (21) is applied:

$$e^m = \frac{\|p^{m+1} - p^m\|}{\|p^m\|} < 10^{-12}. \tag{21}$$

In order to compare CPU times in Fig. 4 and Fig. 5, results of the ADI method without the residual cutting step are used for a normalization of CPU time.

6. NUMERICAL RESULTS

6.1. *Effects of the Number L of α_l^m on Convergence Characteristics*

To explain the typical convergence characteristics in the residual cutting method, the three-dimensional well-posed Neumann problem a-1 in Table I was solved. Figure 3 shows the behavior of relative residual ε^m with the number L of α_l^m as a parameter, taking up the number m of residual cutting steps as the abscissa. Through

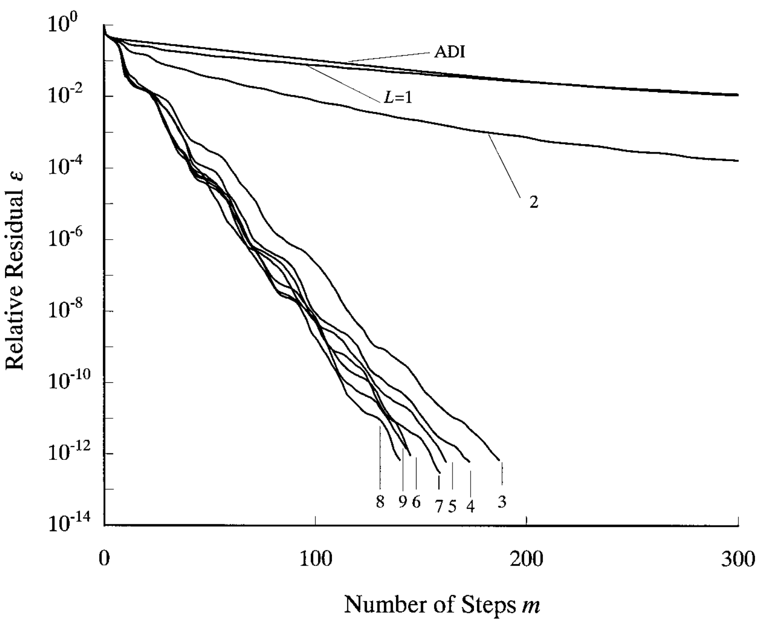


FIG. 3. Effects of the number L of residual cutting coefficients α_l^m on the convergence characteristics (well-posed Neumann problem a-1, $N = 2$).

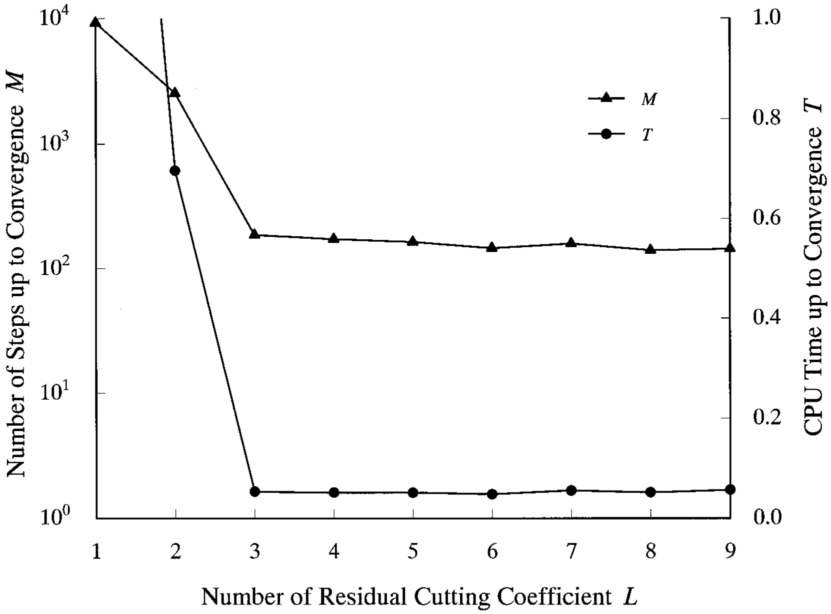


FIG. 4. Optimal number L of residual cutting coefficient α_l^m (well-posed Neumann problem a-1, $N = 2$).

this trial, the number N of inner ADI cycles are fixed as $N = 2$. The ADI method itself is slow in convergence, while the residual cutting method drastically converges when $L \geq 3$. However, it seems that the effects of L become saturated near $L = 3$ or 4 in taking into account the CPU time.

Making clear the effects of the number L of α_l^m, the number M of residual cutting steps and CPU time T up to convergence are plotted in Fig. 4, taking up L as the abscissa. It is seen that $L = 3$ or 4 is sufficient as far as the CPU time is concerned. Almost no difference is noted even if a larger L is taken, and it is rather meaningless to employ too large L because the load of inner product operations becomes greater.

6.2. *Influence of the Number N of Inner ADI Cycles on Convergence Characteristics*

The number N of inner ADI cycles is next discussed in the well-posed Neumann problem a-1. The number N is varied from 1 to 10, fixing the number of α_l^m at $L = 3$. The number M of residual cutting steps and normalized CPU time T up to convergence are plotted in Fig. 5, taking up N as the abscissa. From these results, it is clear that the CPU time does not vary so much according to N, but as a matter of course, M decreases with an increase in N. The optimum number of N concerning CPU time efficiency depends upon the load of inner ADI and the outer residual cutting procedure. So it is possibly affected by the performance of computer and the tuning level of program. However, within the entire test case treated in this paper, $N = 2$ is recommendable as the number of ADI cycles.

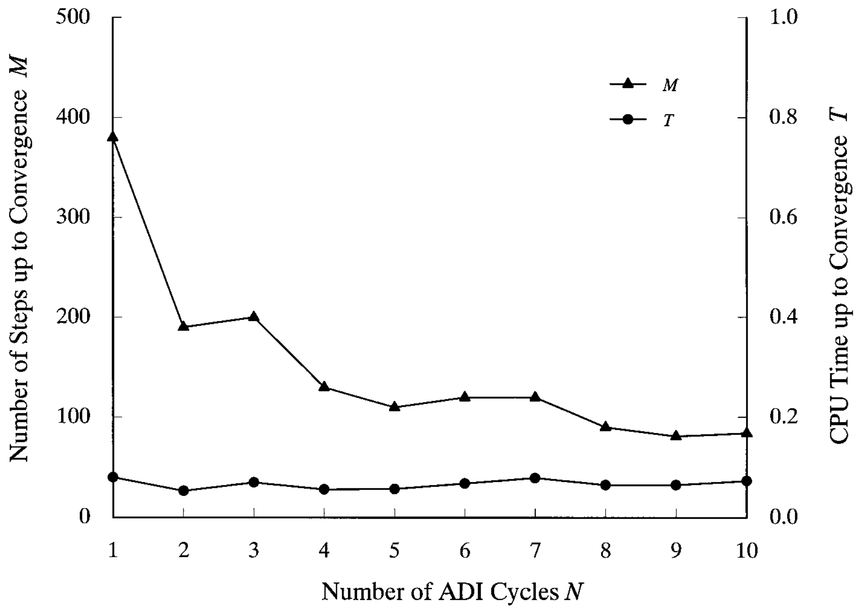


FIG. 5. Optimal number N of inner ADI cycles (well-posed Neumann problem a-1, $L = 3$).

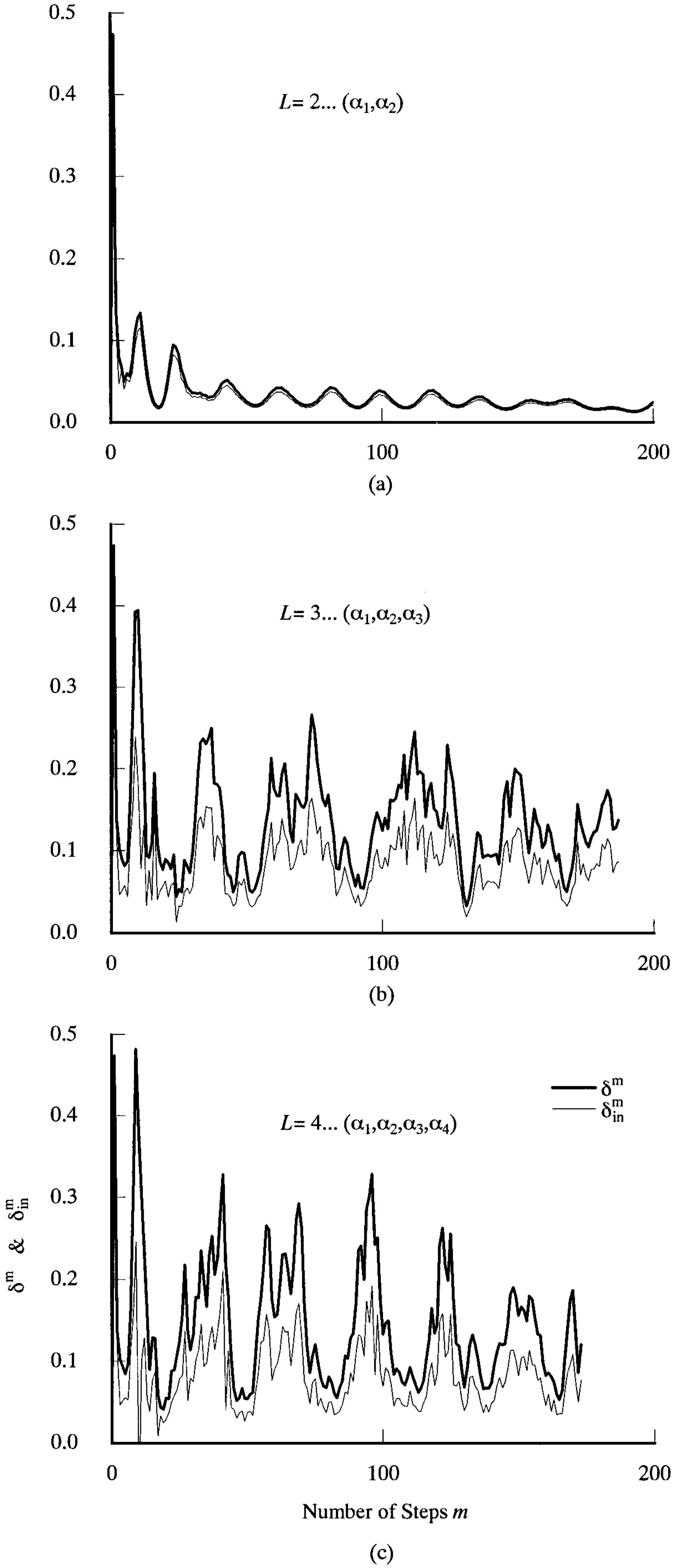
6.3. *Fluctuation of Residual Cutting Ratio* δ^m

The fluctuation of residual cutting ratios δ^m given in (19) and δ^m_{in} given in (20) is shown in Fig. 6 for $N = 2$ and $L = 2, 3, 4$. The solid line and the dotted line signify δ^m and δ^m_{in}, respectively. In general, the residual cutting ratio δ^m becomes larger with the larger L and a higher convergence rate is attained. For example, when $L = 4$ is used as shown in Fig. 6c, the residual cutting ratio δ^m_{in} in inner ADI is almost doubled by the outer residual cutting procedure. This means that the outer residual cutting procedure is very effective in reducing residual. Additionally speaking, the residual cutting ratio δ^m takes as a value $0 < \delta^m \leq 1$, and if δ^m is closer to unity, a higher convergence rate is realized. Due to the load of calculation of residual cutting coefficients, it seems likely that the best result can be expected with $N = 2$ and $L = 3$, considering the CPU time. Results obtained from the test on the two-dimensional Neumann problem also bring about good efficiency for $N = 2$ and $L = 3$, being the same as in the three-dimensional case.

Glancing at Fig. 6, it can be seen that the residual cutting ratio δ^m oscillates. However, it can be imagined that there exists an averaged cutting ratio, which suggests that the large residual rapidly decreases. The residual cutting ratio hereby shown is one of the useful indexes for the convergence rate of a solving method used, when several numerical methods are compared.

6.4. *Convergence Characteristics of an Ill-posed Neumann Problem*

In the next stage, the ill-posed Neumann problem a-2 will be solved using the residual cutting method. The results shown in Fig. 7 are the convergence characteris-



0.5
0.4
0.3
0.2
0.1
0.0
0
100
200
L= 2... (α1,α2)
(a)
L= 3... (α1,α2,α3)
(b)
L= 4... (α1,α2,α3,α4)
δm & δmin
δm
δmin
Number of Steps m
(c)

tics of the problem a-2 with $N = 2$ and $L = 3$. The relative residual converges to its minimum value of 10^{-6}; that is, it does not converge to zero. The result verifies numerically that the source of the problem and boundary conditions of the second kind were set so as to not satisfy the compatibility condition (13). The final nonzero residual represents the degree of ill-posedness of the problem. On the other hand, the relative error of pressure p converges up to the convergence criterion of 10^{-12}. Comparing the well-posed problem a-1 and ill-posed problem a-2, it is seen that the latter converges faster. Incidentally, according to Eq. (14), if the final nonzero residual in the problem a-2 is subtracted from the right-hand side in a-2, and it is solved again, one can obtain the same solution with a zero final residual. In general engineering applications, it is not necessarily easy to set Neumann problems as well-posed problems. In any case, it will be understood that the residual cutting method has favorable characteristics suitable for actual engineering applications.

6.5. *Boundary Condition Dependency*

The convergence characteristics of the Neumann problem a-1 and Dirichlet problem a-3 for the pressure distribution in a curved duct are shown in Fig. 8. The residual cutting method is also valid for the Dirichlet problem for which it attains a convergence rate three times higher than that of the Neumann problem. Generally, Dirichlet problems are easier to solve than Neumann problems, and the convergence rate of mixed boundary value problems shows the intermediate characteristics between the former and the latter, according to the number of boundary conditions of the second kind.

6.6. *Initial Value Dependency*

In order to examine robustness, the initial value p^0 at interior grid points in all the test problems is set to be zero as the most rough initial value. However, the residual cutting method is stable even if such a rough initial value is provided. Usually in Navier–Stokes computation, the convergence of Poisson's equation is drastically enhanced because of the fact that its initial value is improved according to the progress of computation.

6.7. *Computational Grid Dependency*

To show the general validity of the residual cutting method, let us examine the convergence in the Neumann problems b-1, b-2, Dirichlet problem b-3, and mixed boundary value problem b-4 of Poisson's equation described in the general curvilinear coordinate system. In each case, $N = 2$ is used as the number of inner ADI cycles and $L = 5$ is used as the number of residual cutting coefficients, the reason for which will be referred to later.

Figure 9 shows the convergence characteristics of relative residual and relative

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FIG. 6. Fluctuation of residual cutting ratio.

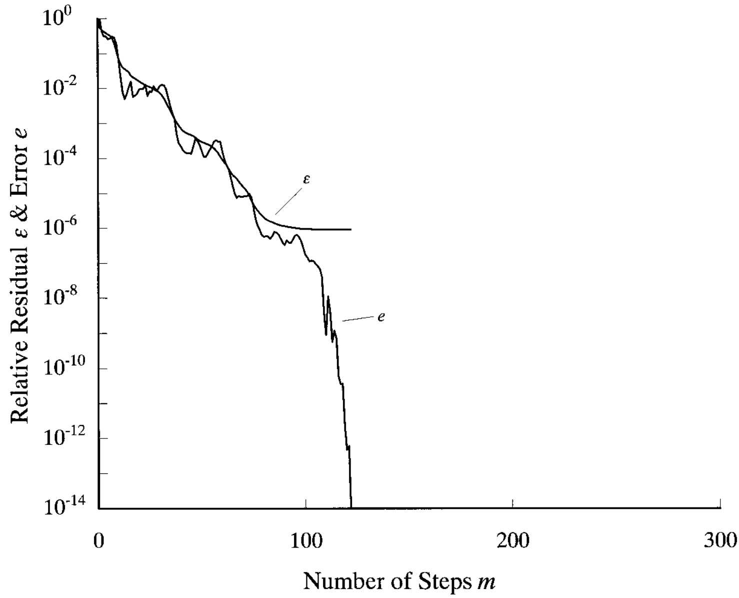

FIG. 7. Convergence characteristics of the ill-posed Neumann problem a-2 ($N = 2$, $L = 3$): curved duct.

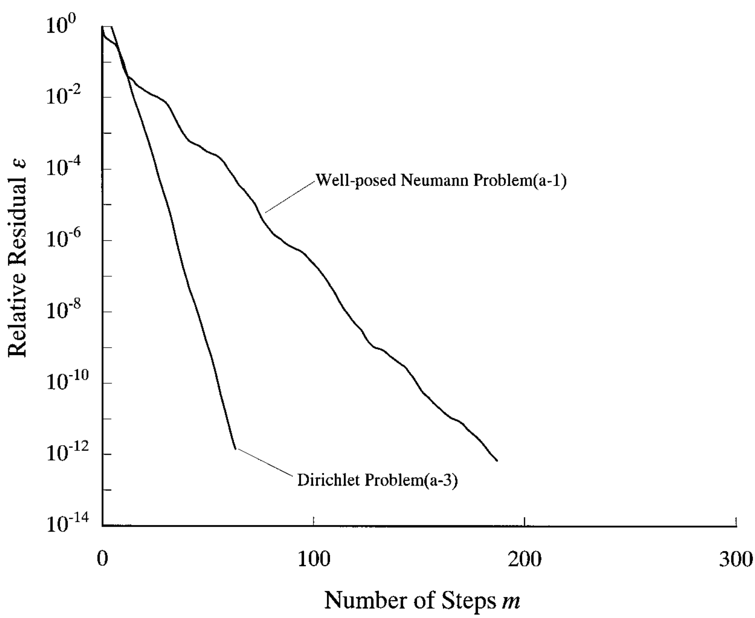

FIG. 8. Convergence characteristics of the well-posed Neumann problem a-1 and the Dirichlet problem a-3 ($N = 2$, $L = 3$): curved duct.

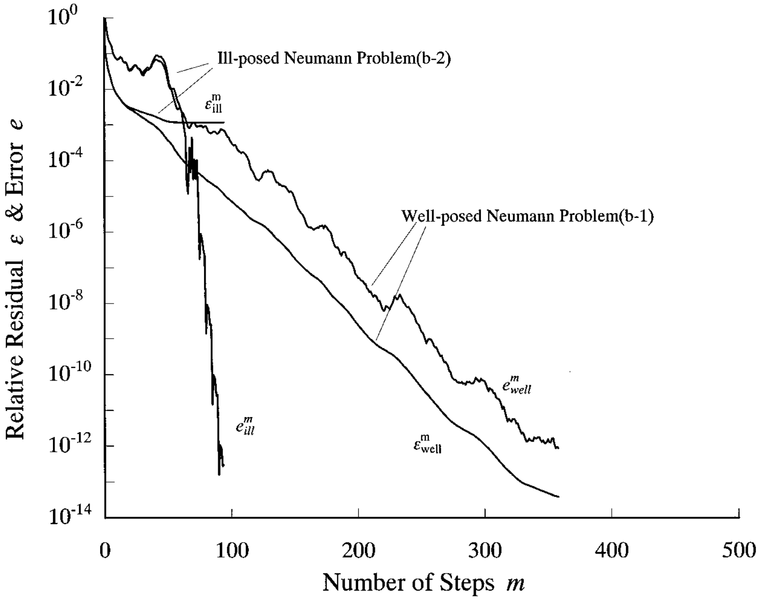

FIG. 9. Convergence characteristics of the well-posed Neumann problem b-1 and the ill-posed Neumann problem b-2 ($N = 2$, $L = 5$): cascade.

error in the well-posed Neumann problem b-1 and ill-posed Neumann problem b-2. The relative residual $\varepsilon^m_{\mathrm{well}}$ and relative error e^m_{well}, in case of the well-posed problem b-1, reach to the target points. On the other hand, in the case of the ill-posed problem b-2, the relative residual $\varepsilon^m_{\mathrm{ill}}$ does not converge to zero, while the relative error, e^m_{ill} rapidly converges. It is the same as in the case of the orthogonal curvilinear coordinate system.

Comparing ε^m of the well-posed problem a-1 in the orthogonal curvilinear coordinate system (see Fig. 8) and $\varepsilon^m_{\mathrm{well}}$ of the well-posed problem b-1 in the general curvilinear coordinate system, it is seen that the number of steps for the latter is approximately twice that of the former. However, in ill-posed problems, it is seen that the difference between the orthogonal curvilinear coordinate system and the general curvilinear coordinate system is very small from the viewpoint of general engineering applications.

The convergence characteristics of relative residual ε^m and relative error e^m in the Dirichlet problem b-3 are shown in Fig. 10. The same rapid convergence is noted with both of them, and the convergence is almost never inferior to the problem a-3 in the orthogonal curvilinear coordinate system (see Fig. 8). Comparing the Neumann problem with the Dirichlet problem, a considerable degree of difference is seen in the convergence characteristics. In fact, it will be understood that the convergence in the Dirichlet problem is generally several times more rapid than that in the Neumann problem. It is the same in the case of the orthogonal curvilinear coordinate system.

The convergence characteristics in the mixed boundary value problem b-4 with two surfaces having boundary conditions of the first kind and the other four surfaces

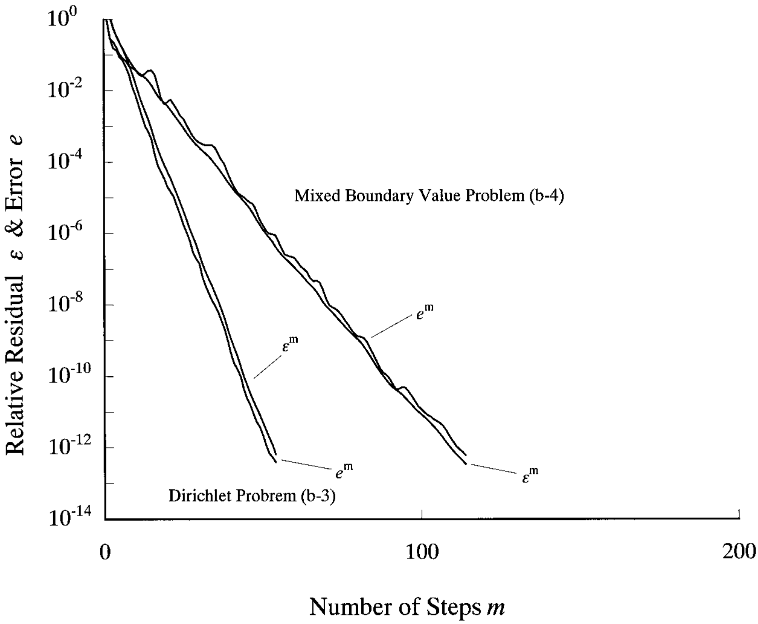

FIG. 10. Convergence characteristics of the Dirichlet problem b-3 and the mixed boundary value problem b-4 ($N = 2$, $L = 5$): cascade.

having boundary conditions of the second kind is also shown in Fig. 10. How the convergence is made indicates the intermediate characteristics between the Dirichlet problem and the well-posed Neumann problem. In terms of condition number, this is expressed as

$$\|A_{\mathrm{D}}\|\,\|A_{\mathrm{D}}^{-1}\| < \|A_{\mathrm{M}}\|\,\|A_{\mathrm{M}}^{-1}\| < \|A_{\mathrm{N}}\|\,\|A_{\mathrm{N}}^{-1}\|,$$

where A_{D} denotes the coefficient matrix corresponding to the Dirichlet problem and so on, and A_{N} is regarded as an operator on R^{n-1}.

As mentioned above, $L = 5$ was adopted as the number of residual cutting coefficients for the problems b-1, b-2, b-3, and b-4 in the general curvilinear coordinate system. It has been found from identical tests with those in the orthogonal curvilinear coordinate system that $L = 5$ or 6 is enough both for the number of residual cutting steps and more importantly for CPU time. Such requirement for L seems to be owing to the treatment of difference terms corresponding to cross-derivatives as mentioned in Section 5.

6.8. *Number of Grid Points Dependency*

In order to show how the residual cutting method behaves as the number of grid points is increased, the Dirichlet and well-posed Neumann problems for a cascade are examined varying the number of the grid points. The fine and coarse grids made from original one retaining the physical domain.

It seems that the number M of steps up to convergence ($e \le 10^{-12}$) becomes

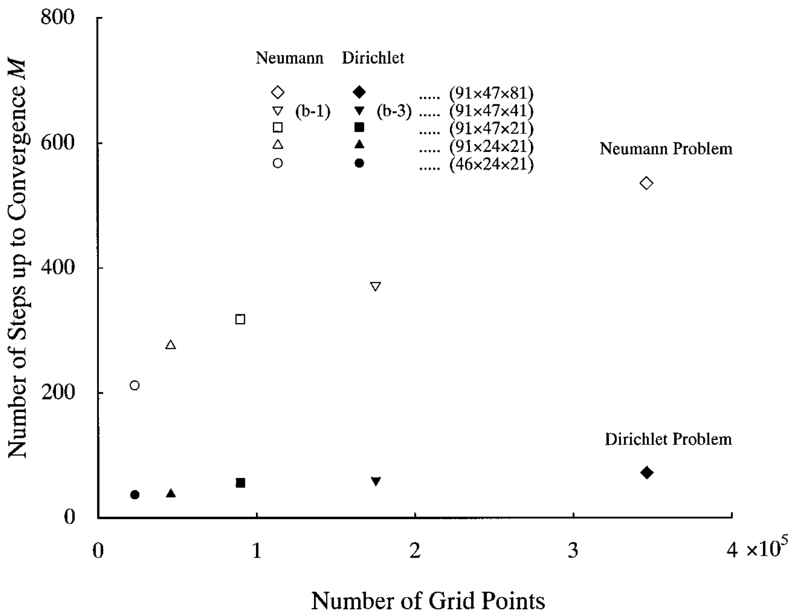

FIG. 11. Multigrid-like property of residual cutting method: cascade.

independent of the number of the grid points both for Dirichlet and Neumann problems (see Fig. 11). The multigrid-like property of the Dirichlet problem is established at a relatively small number of grid points, compared with that of Neumann problems. The multigrid-like property confirms the rapid decrease of the large residual as mentioned in Subsection 6.3. Thus the residual cutting method has a multigrid-like property and this is quite advantageous for actual engineering applications.

7. CONCLUSIONS

The residual cutting method is proposed as a numerical method for boundary value problems of elliptic partial differential equations. The residual equation is set and its approximate solution is obtained within the first few iterations. The desired perturbed quantity is detained by the linear combination of the approximate solution and previous perturbed quantities. The coefficients, called residual cutting coefficients, are determined by the method of least squares so as to reduce the residual L2 norm. The residual equation is updated and the above procedure is repeated up to convergence.

The residual cutting method is quite simple, but provides with a high convergence rate and a high level of robustness for various types of boundary value problems including Neumann problems. It is recognized that Neumann problems remain a difficulty, especially in general curvilinear coordinate system. However, in our solving method, Neumann problems in curvilinear coordinate systems are successfully dealt with in the same manner as Dirichlet problems. The mechanism of residual reduction is nothing but the fact that the inner solver implicitly decreases the residual and the outer residual cutting step explicitly decreases the residual. The ADI method is used as an inner solver of the residual equation and no special acceleration is done. Accordingly, if adequate acceleration parameters are employed numerical results may be improved to a higher convergence rate.

By numerical computations of three-dimensional Poisson's equation for pressure, the following results were obtained:

(1) The residual cutting method can, despite its simple algorithm, solve elliptic partial differential equations steadily, with high accuracy and at a high rate of convergence.

(2) The residual cutting method does not necessarily require the so-called compatibility condition which is a major difficulty in solving Neumann problem. In fact, it is shown that the target solution of an ill-posed Neumann problem is same as that of the corresponding well-posed Neumann problem which is reasonably preset.

(3) The residual cutting method compensates for the ADI method, because the ADI method itself is not necessarily available for three-dimensional problems in general curvilinear coordinate systems.

(4) The number N of inner ADI cycles is enough with $N = 2$. Also, the number L of residual cutting coefficients is enough with $L = 3$ in the orthogonal curvilinear grid system and with $L = 5$ in the general curvilinear grid system. From a general point of view, $N = 2$ and $3 \leq L \leq 6$ are recommended and it is not required that N and L are taken any larger.

(5) In terms of CPU time, the Dirichlet problem most rapidly converges, followed by the mixed boundary value problem, with the Neumann problem being the slowest to reach convergence.

(6) There exists no dependence on initial values, and steady computation is carried out from a roughly approximated initial value.

(7) The residual cutting method has a multigrid-like property and this is quite advantageous for actual engineering applications.

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